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A FINITE ELEMENT METHOD FOR THE THREE-DIMENSIONAL NON-STEADY NAVIER–STOKES EQUATIONS

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SUMMARY

The algorithm for solving the three-dimensional non-steady Navier–Stokes equations by the explicit forward Euler method is shown and the Galerkin finite element formulation is presented. As a numerical example, an entrance flow in a square duct is illustrated.

KEY WORDS Finite Elements Non-steady Flow Fractional Step Method

INTRODUCTION

Recently, finite element methods (FEM) based on a fractional step approach have been presented for three-dimensional (and/or large, two-dimensional) time-dependent flow problems.[1–5] They are an application of Chorin's method[6,7] in finite difference formats to FEM and have a simple algorithmic structure of treating the momentum equations and the continuity equation separately. The discretization, however, is rather complicated for the simplicity of the algorithm, because the essential boundary condition for the normal component of the velocity must be embodied into the discretized continuity equation.[2]

In the present paper, we show a new version of the fractional step method. In our method, to eliminate the above complexity a potential function is introduced and Poisson equation for the potential is directly discretized, in which the essential boundary condition for the normal component of the velocity is treated as the natural boundary condition for the potential. This method is also applicable to steady flow problems.[8]

As a numerical example, an entrance flow in a square duct will be illustrated in the last section.

ALGORITHM

The non-steady Navier–Stokes equations and the continuity equation written in non-dimensional form are:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u} \tag{1}$$

$$\nabla.\mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}$, p, t and Re are velocity vector, pressure, time and the Reynolds number, respectively, and the other mathematical symbols are used in the standard manner.

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Our method is a kind of time marching scheme and is based on Helmholtz's decomposition theorem. It is formulated as follows:

Letting $\mathbf{u}^m$ and p^m be the solution at $t = m\,\delta t$ (δt is a time increment), we define $\mathbf{u}^{m+1}$ and p^{m+1} through:

$$\frac{\mathbf{u}^{m+1}-\mathbf{u}^m}{\delta t}+(\mathbf{u}^m.\nabla)\mathbf{u}^m=-\nabla p^{m+1}+\frac{1}{Re}\nabla^2\mathbf{u}^m \tag{3}$$

$$\nabla.\mathbf{u}^{m+1}=0 \tag{4}$$

Equation (3) is rewritten as follows:

$$\mathbf{u}^{m+1}+\delta t\,\nabla p^{m+1}=\mathbf{u}^m+\delta t\left\{\frac{1}{Re}\nabla^2\mathbf{u}^m-(\mathbf{u}^m.\nabla)\mathbf{u}^m\right\} \tag{5}$$

The left-hand side of equation (5) is Helmholtz's decomposition of the right-hand side because $\mathbf{u}^{m+1}$ is solenoidal ($\nabla.\mathbf{u}^{m+1}=0$) and ∇p^{m+1} is irrotational ($\nabla\times\nabla p^{m+1}=0$).

Therefore, the procedure of Helmholtz's decomposition is the key to our method. Such a decomposition can be done as follows:

First, we define $\mathbf{u}^{m+\frac{1}{2}}$ by:

$$\mathbf{u}^{m+\frac{1}{2}}=\mathbf{u}^m+\delta t\left\{-\nabla p^m+\frac{1}{Re}\nabla^2 u^m-(\mathbf{u}^m.\nabla)\mathbf{u}^m\right\} \tag{6}$$

In general, $\mathbf{u}^{m+\frac{1}{2}}$ is not solenoidal because p^m is used instead of p^{m+1} in equation (5).

Taking the rotation of equations (5) and (6) leads to:

$$\nabla\times\mathbf{u}^{m+1}=\nabla\times\mathbf{u}^{m+\frac{1}{2}} \tag{7}$$

Equation (7) means:

$$\mathbf{u}^{m+1}=\mathbf{u}^{m+\frac{1}{2}}+\nabla\phi \tag{8}$$

where ϕ is some potential function.

Taking the divergence of equation (8) and using equation (4) leads to:

$$\nabla^2\phi=-\nabla.\mathbf{u}^{m+\frac{1}{2}} \tag{9}$$

We can obtain the potential ϕ from equation (9) and then the velocity $\mathbf{u}^{m+1}$ from equation (8).

On the other hand, substituting equations (5) and (6) into equation (8) leads to:

$$\delta t\,\nabla p^{m+1}=\delta t\,\nabla p^m-\nabla\phi$$

i.e.

$$p^{m+1}=p^m-\frac{\phi}{\delta t} \tag{10}$$

where an integral constant is set equal to zero.

The above procedure is summarized in Figure 1.

INITIAL AND BOUNDARY CONDITIONS

Let Ω be the flow region to be solved and Γ be the boundary of Ω.

The initial conditions are given as follows:

$$\begin{aligned}\mathbf{u}(\mathbf{x},0)&=\mathbf{u}^0(\mathbf{x})\\ p(\mathbf{x},0)&=p^0(\mathbf{x})\end{aligned} \tag{11}$$

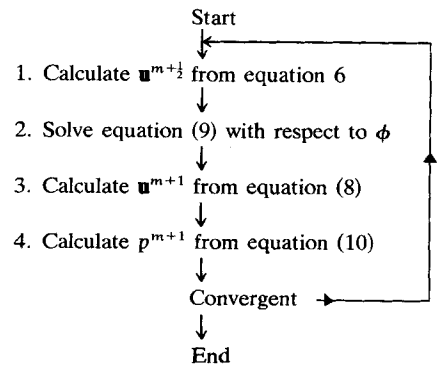

Figure 1. Algorithm

In the present method, the initial velocity field $\mathbf{u}^0$ does not always have to satisfy the incompressibility condition.

In the velocity–pressure formulations, the surface traction is usually given as the natural boundary condition. It is, however, difficult to visualize such a condition in flow problems.

In the present paper, we therefore adopt the following boundary conditions.[9]

$$\begin{aligned} \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma_1 \\ -p\mathbf{n} + \frac{1}{Re}\frac{\partial \mathbf{u}}{\partial n} &= \mathbf{h} \quad \text{on } \Gamma_2 \end{aligned} \tag{12}$$

where $\mathbf{g}$ and $\mathbf{h}$ are given boundary data, $\mathbf{n}$ is the unit vector outward normal to the boundary, $\partial/\partial n$ is the outward normal derivative to the boundary, and Γ_1 and Γ_2 are subsets of Γ satisfying the following conditions:

$$\begin{aligned} \overline{\Gamma_1 \cup \Gamma_2} &= \Gamma \\ \Gamma_1 \cap \Gamma_2 &= \emptyset \end{aligned} \tag{13}$$

In our method, there also must be given the boundary conditions for ϕ. We assume that they are expressed as follows:

$$\begin{aligned} \frac{\partial \phi}{\partial n} &= 0 \quad \text{on } \Gamma_1 \\ \phi &= 0 \quad \text{on } \Gamma_2 \end{aligned} \tag{14}$$

From equations (8) and (10), we can see that the boundary conditions (14) mean:

$$\begin{aligned} \mathbf{n}.\mathbf{u}^{m+1} &= \mathbf{n}.\mathbf{u}^{m+\frac{1}{2}} \quad \text{on } \Gamma_1 \\ p^{m+1} &= p^m \quad \text{on } \Gamma_2 \end{aligned} \tag{15}$$

It should be noticed that the boundary condition $(15)_1$ is for the normal component of the velocity to the boundary and that it says nothing about the tangential component. In general, the tangential component of $\mathbf{u}^{m+1}$ is different from that of $\mathbf{u}^{m+\frac{1}{2}}$ and this difference decreases as $\mathbf{u}^{m+1}$ converges to the steady solution.

GALERKIN FORMULATION

In this section, we use the 8-node isoparametric brick, in which the velocity and the potential are interpolated by trilinear basis functions N_k and the pressure is piecewise constant (defined at the centroid of each element).

Letting (u, v, w) be the velocity components in an orthogonal Cartesian co-ordinate system (x, y, z), equation (6) is rewritten as follows:

$$\begin{aligned}
u^{m+\frac{1}{2}} &= u^m + \delta t\left\{-\frac{\partial p^m}{\partial x} + \frac{1}{Re}\nabla^2 u^m - \left(u^m\frac{\partial u^m}{\partial x} + v^m\frac{\partial u^m}{\partial y} + w^m\frac{\partial u^m}{\partial z}\right)\right\} \\
v^{m+\frac{1}{2}} &= v^m + \delta t\left\{-\frac{\partial p^m}{\partial y} + \frac{1}{Re}\nabla^2 v^m - \left(u^m\frac{\partial v^m}{\partial x} + v^m\frac{\partial v^m}{\partial y} + w^m\frac{\partial v^m}{\partial z}\right)\right\} \\
w^{m+\frac{1}{2}} &= w^m + \delta t\left\{-\frac{\partial p^m}{\partial z} + \frac{1}{Re}\nabla^2 w^m - \left(u^m\frac{\partial w^m}{\partial x} + v^m\frac{\partial w^m}{\partial y} + w^m\frac{\partial w^m}{\partial z}\right)\right\}
\end{aligned} \tag{16}$$

Applying the conventional Galerkin method to each of equations (16) and the boundary conditions (12) leads to the element matrix equations:

$$\begin{aligned}
M_{ij}u_j^{m+\frac{1}{2}} &= M_{ij}u_j^m + \delta t\left(B_{xi} + p_e^m C_{xi} - \frac{1}{Re}A_{ij}u_j^m - D_{ij}^m u_j^m\right) \\
M_{ij}v_j^{m+\frac{1}{2}} &= M_{ij}v_j^m + \delta t\left(B_{yi} + p_e^m C_{yi} - \frac{1}{Re}A_{ij}v_j^m - D_{ij}^m v_j^m\right) \\
M_{ij}w_j^{m+\frac{1}{2}} &= M_{ij}w_j^m + \delta t\left(B_{zi} + p_e^m C_{zi} - \frac{1}{Re}A_{ij}w_j^m - D_{ij}^m w_j^m\right)
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
M_{ij} &= \int_{\Omega_e} N_i N_j \,\mathrm{d}\Omega_e \\
A_{ij} &= \int_{\Omega_e}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\right)\mathrm{d}\Omega_e \\
B_{\alpha i} &= \int_{\Gamma_e} N_i h_\alpha \,\mathrm{d}\Gamma_e \quad (\alpha = x, y, z) \\
C_{\alpha i} &= \int_{\Omega_e}\frac{\partial N_i}{\partial \alpha}\,\mathrm{d}\Omega_e \quad (\alpha = x, y, z)
\end{aligned} \tag{18}$$

$$D_{ij}^m = \int_{\Omega_e} N_i\frac{\partial N_j}{\partial x}N_k u_k^m \,\mathrm{d}\Omega_e + \int_{\Omega_e} N_i\frac{\partial N_j}{\partial y}N_k v_k^m \,\mathrm{d}\Omega_e + \int_{\Omega_e} N_i\frac{\partial N_j}{\partial z}N_k w_k^m \,\mathrm{d}\Omega_e$$

In the above equations, indices i, j and k denote the element node number; the summation convention is employed on repeated indices (in this case, 1 to 8), index e denotes the element number, and (h_x, h_y, h_z) denote the Cartesian components of $\mathbf{h}$.

For saving CPU time, we adopted two corruptions:

1. The element matrix M_{ij} is the so-called 'mass matrix' and we simplify it to the 'lumped mass' approximation by row-sum at element level.
2. The element matrix D_{ij}^m (advection matrix) is modified as follows.[4]

$$D_{ij}^m = u_e^m E_{xij} + v_e^m E_{yij} + w_e^m E_{zij} \tag{19}$$

where

$$u_e^m = \frac{1}{\text{vol}\,(\Omega_e)} \int_{\Omega_e} N_k u_k^m \, d\Omega_e; \quad \text{etc.}$$
$$E_{\alpha ij} = \int_{\Omega_e} N_i \frac{\partial N_j}{\partial \alpha} \, d\Omega_e \quad (\alpha = x, y, z) \tag{20}$$

Applying the conventional Galerkin method to equation (9) and the boundary conditions (14) leads to the element matrix equation:

$$A_{ij}\phi_j = E_{xij}u_j^{m+\frac{1}{2}} + E_{yij}v_j^{m+\frac{1}{2}} + E_{zij}w_j^{m+\frac{1}{2}} \tag{21}$$

It should be noticed that the element matrix A_{ij} is symmetric and constant at any time step. Therefore once we solve the global equations by using Gauss elimination, the global matrix of A_{ij} is factored into so-called $\mathbf{LDL}^{\mathrm{T}}$ and stored on disk (or core memory).

Applying the conventional Galerkin method to equation (8) leads to the element matrix equations:

$$\begin{aligned} M_{ij}u_j^{m+1} &= M_{ij}u_j^{m+\frac{1}{2}} + E_{xij}\phi_j \\ M_{ij}v_j^{m+1} &= M_{ij}v_j^{m+\frac{1}{2}} + E_{yij}\phi_j \\ M_{ij}w_j^{m+1} &= M_{ij}w_j^{m+\frac{1}{2}} + E_{zij}\phi_j \end{aligned} \tag{22}$$

This element matrix M_{ij} is also lumped.

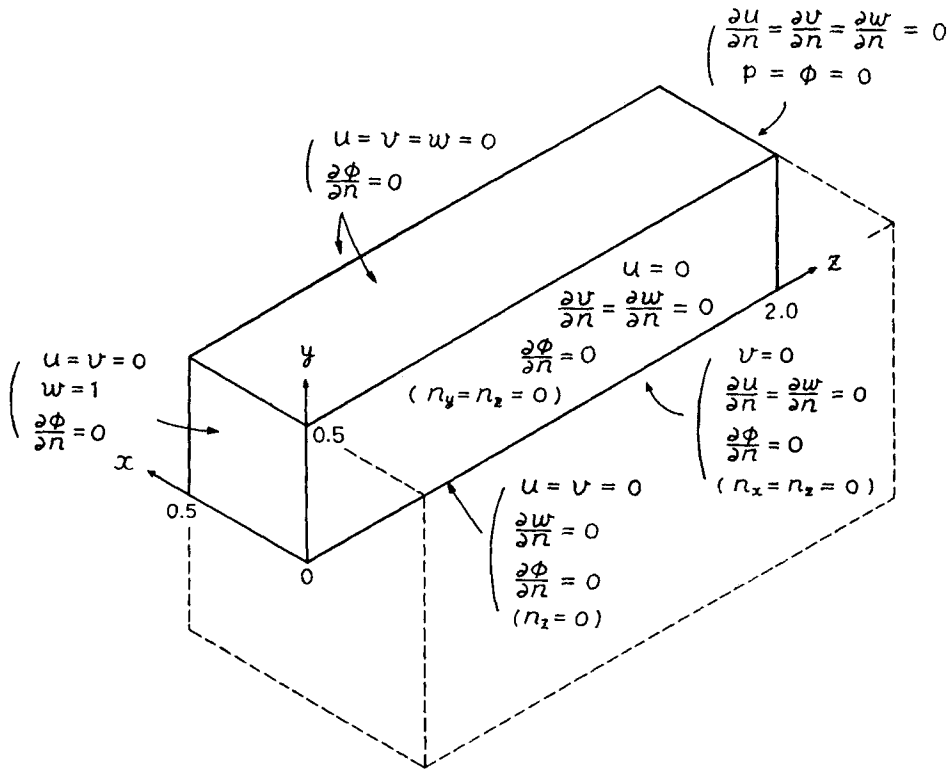

Figure 2. Boundary conditions

As for equation (10), the piecewise linear function ϕ is reduced to the piecewise constant pressure p as follows:

$$p_e^{m+1} = p_e^m - \frac{\phi_e}{\delta t} \tag{23}$$

where ϕ_e denotes the value at the centroid of element e and is expressed as the same as u_e^m etc.

NUMERICAL EXAMPLE

Numerical computation was done for an entrance flow in a square duct. The boundary conditions and the finite element mesh are shown in Figures 2 and 3, respectively. Only a quarter part is calculated in consideration of the symmetricity of the duct. The initial conditions are $\mathbf{u} = \mathbf{0}$ and $p = 0$. The Reynolds number is 20, which is calculated with respect to the side length of the duct and the inlet velocity.

Figure 4 shows the steady state pressure contour lines. We can see the large pressure gradient at the inlet. In Figures 5, 6 and 7, we show the steady state velocity distributions.

CONCLUSIONS

This paper has presented the new version of the finite element method based on the fractional step approach for the three-dimensional non-steady Navier–Stokes equations. As a numerical example, the entrance flow in a square duct was illustrated.

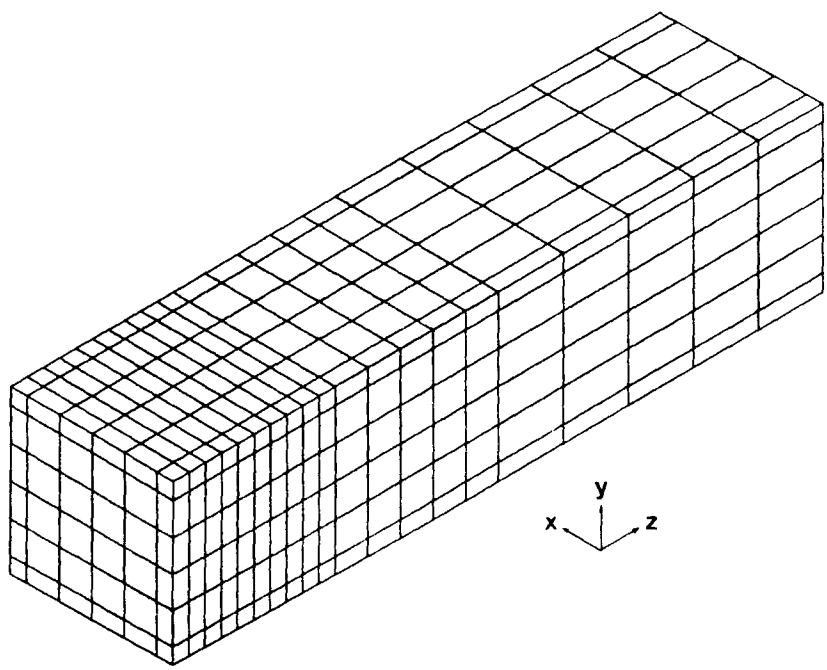

Figure 3. Finite element mesh

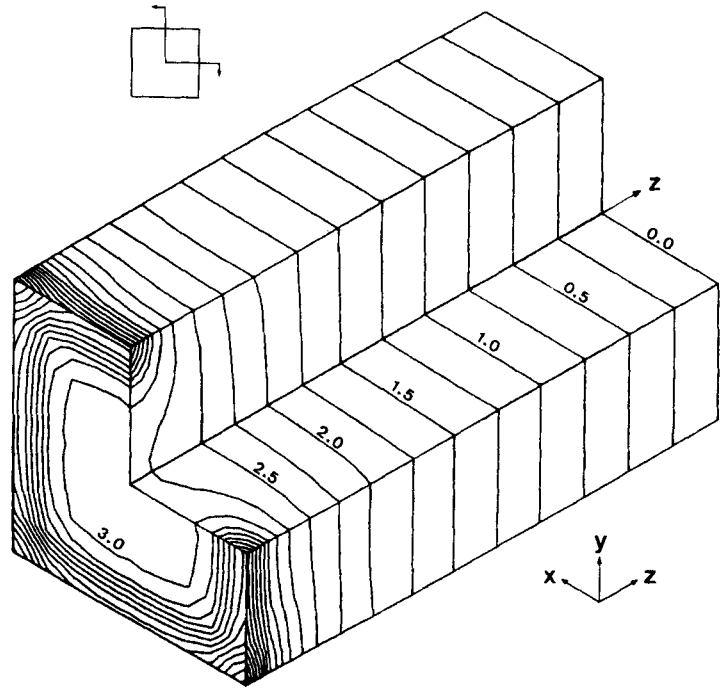

Figure 4. Steady state pressure contour lines

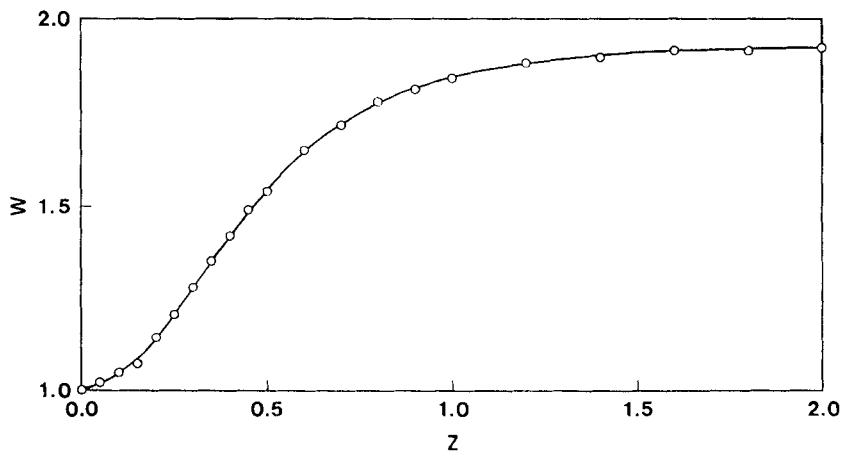

Figure 5. Steady state axial velocity development at the duct centre line

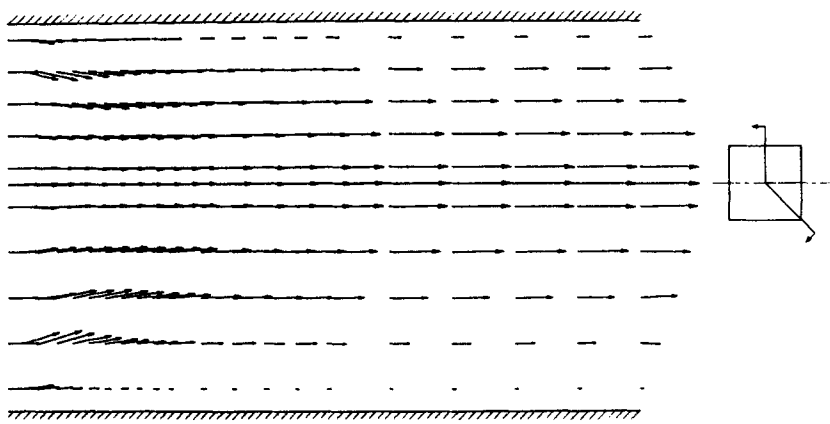

Figure 6. Steady state velocity vectors on the symmetric cross-sections

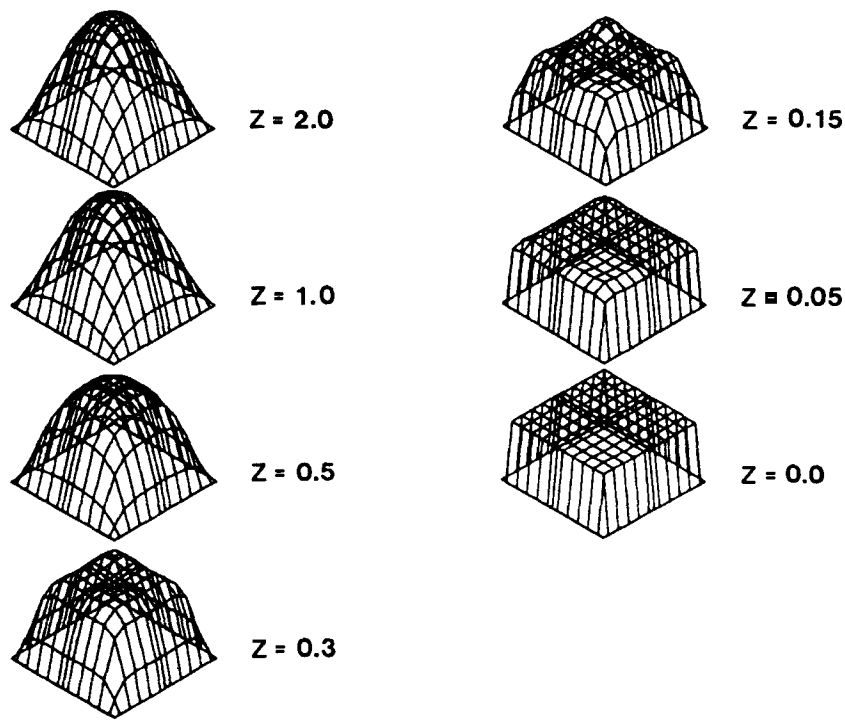

Figure 7. Three-dimensional view of steady state velocity distribution

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